A LESSON ABOUT TESSELLATIONS

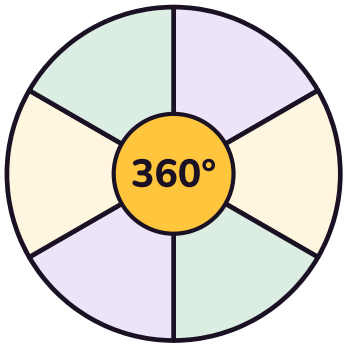
The artist who failed maths ended up inspiring mathematicians

A museum in New York is celebrating the artist M.C. Escher. As a student he struggled with maths, and later said he "never got a pass mark." His tilings still inspire mathematicians. Today you will find out why some shapes tile a floor with no gaps — and why regular pentagons do not.

Name _____ **Class / year** _____ **Date** _____

1. Angles at a point

To tile a floor with no gaps and no overlaps, the angles that meet at a point must add to a full turn: **360°**.



Angles at a point add to 360°

Q1. What is the sum of the angles that meet at a point?

They add to _____ °.

Q2. Why does this matter if you want to tile a floor?

2. Interior angles of regular polygons

A **regular** polygon has all sides equal and all interior angles equal. The interior angle of a regular n-sided polygon is

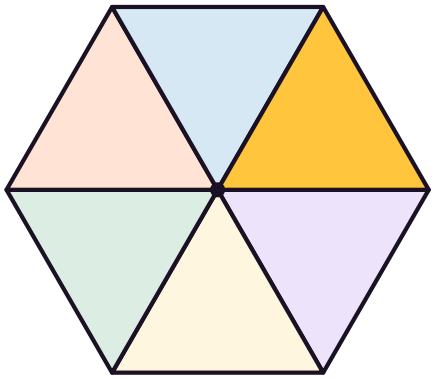
$$\frac{(n - 2) \times 180°}{n}$$

Q3. Complete the table. Show your working in the middle column.

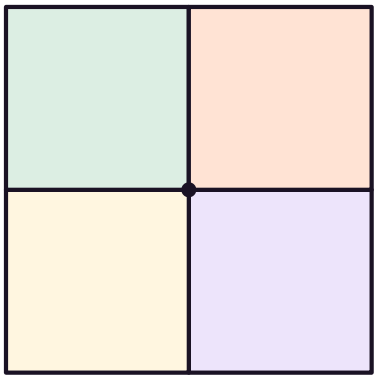
Regular polygon	n	Working	Interior angle
Equilateral triangle	3		
Square	4		
Regular pentagon	5		
Regular hexagon	6		
Regular octagon	8		

3. Which regular polygons tessellate?

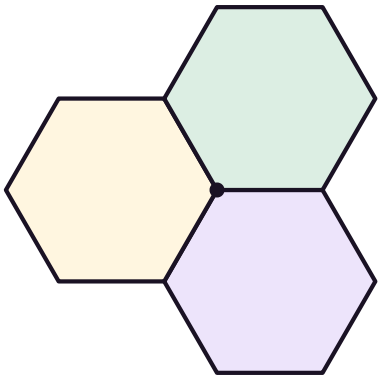
A regular polygon **tessellates** — it tiles the plane by itself — if a whole number of copies can meet at a point with no gap and no overlap. That means the interior angle must divide 360°, and at least three polygons must meet at each point.



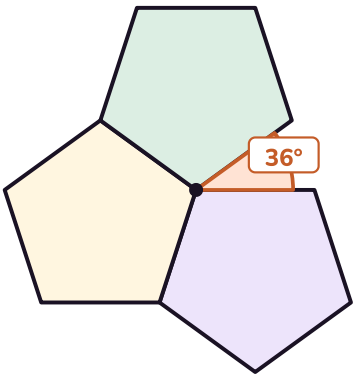
60° × 6 = 360°



90° × 4 = 360°



120° × 3 = 360°



108° × 3 = 324°, so a 36° gap remains

Q4. How many equilateral triangles fit around a point? Complete: 60° × _____ = 360°.

Q5. How many squares fit around a point? 90° × _____ = 360°.

Q6. How many regular hexagons fit around a point? 120° × _____ = 360°.

Q7. A regular pentagon has an interior angle of 108°. Work out 108° × 3, then find the gap that is left if you try to fit three around a point.

Gap = 360° − _____° = _____°. Does 108 divide 360? _____

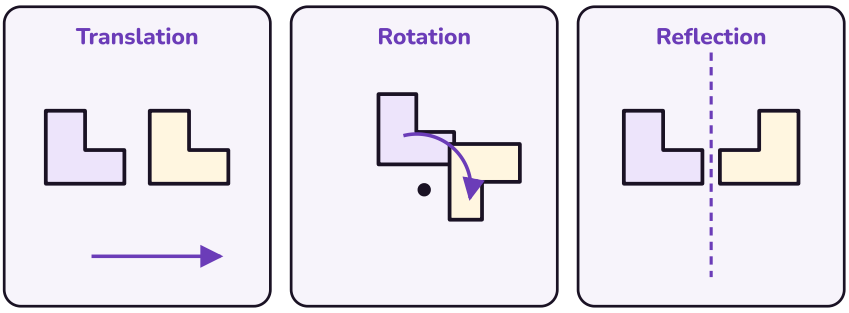
Q8. Can regular octagons tile a floor by themselves? Use your table. (Try 135° × 2 and 135° × 3.)

Q9. A friend says “any regular polygon will tile a floor if you use enough of them.” How would you answer?

> The only regular polygons that tessellate by themselves are equilateral triangles, squares and regular hexagons. Regular pentagons leave a gap.

4. Slide, turn or flip

You can also move a shape without changing its size or its angles. The three transformations below are a **translation** (a slide), a **rotation** (a turn about a point) and a **reflection** (a flip over a mirror line).

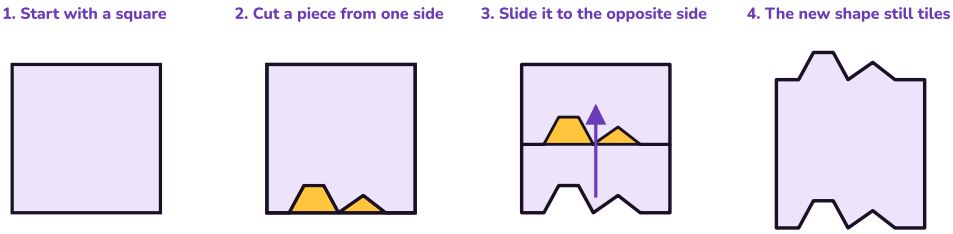


Q10. In your own words, what is the difference between a translation and a rotation?

Q11. A motif is flipped over a vertical line. Which of the three transformations is that?

5. Make your own tile

Here is a classic way to invent an Escher-style tile. All the diagrams on this worksheet are original — they are not copies of Escher's prints, and this is a classroom method, not a claim that he invented this exact construction.



1. Start with a square.
2. Cut a piece from one side. (A zigzag or a simple bump is fine — invent your own.)
3. Slide that piece, without turning it, to the opposite side and attach it there. That slide is a translation.
4. The new shape has the same area as the square, and it tiles the plane by translation: when you slide it by one side-length, the bump fills the hole.

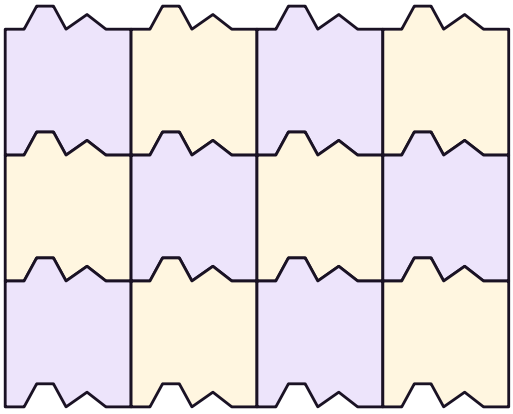
Q12. Why does the new shape still tile, with no gaps?

Q13. Does the new tile have the same area as the square you started with? Explain.

6. Design your tile, then see it repeat

Please do not cut this worksheet. Sketch your design in the square, then cut and slide on spare paper (or a printed square from your teacher). Colour two copies if you can, and check that they fit together with no gaps.

My cut-and-slide tile



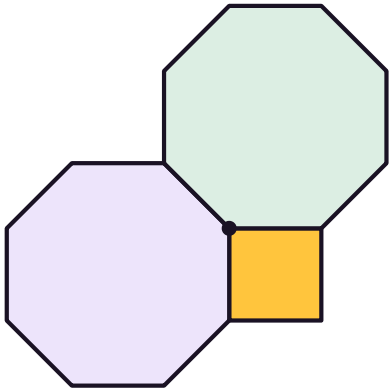
It still tiles — no gaps

An example (the zigzag is just one idea — invent your own). It still tiles, in two colours, with no gaps.

7. Extension — semi-regular tessellations

A **semi-regular** (or Archimedean) tessellation uses *two or more* kinds of regular polygon, with the *same* arrangement at every vertex. The angles at a vertex must still add to 360°.

E1. A square and two regular octagons meet at a point. Using your table, check that the angles add to 360°. This arrangement is written 4.8.8.



A square and two regular octagons

E2. Find another mix of regular polygons whose interior angles add to 360°. Write the angles, and give an example (two that work are 60° + 150° + 150°, and 60° + 120° + 60° + 120°).

E3. Why are equilateral triangles, squares and regular hexagons the *only* regular tessellations? You can either check the interior angle for n = 3, 4, 5, 6, 7 and 8, or show that $2n / (n - 2)$ must be a whole number of at least 3, so $n - 2$ must be a positive factor of 4.

A note for the curious: some *irregular* pentagons can tessellate. Today's claim is only about regular ones — all sides equal, all angles equal.

Answers — for teachers, or for checking afterwards

Q1. 360°.

Q2. If the angles at a point add to less than 360°, a gap is left. If they add to more, the shapes overlap. Accept any clear version of "they must make a full turn, with no gap and no overlap."

Q3. Interior angles:

Regular polygon	*n*	Working	Interior angle
Equilateral triangle	3	(3 − 2) × 180° / 3 = 180° / 3	**60°**
Square	4	(4 − 2) × 180° / 4 = 360° / 4	**90°**
Regular pentagon	5	(5 − 2) × 180° / 5 = 540° / 5	**108°**
Regular hexagon	6	(6 − 2) × 180° / 6 = 720° / 6	**120°**
Regular octagon	8	(8 − 2) × 180° / 8 = 1,080° / 8	**135°**

Q4. 6, because 60° × 6 = 360°.

Q5. 4, because 90° × 4 = 360°.

Q6. 3, because 120° × 3 = 360°.

Q7. 108° × 3 = **324°**. Gap = 360° − 324° = **36°**. 108 does not divide 360 (360 ÷ 108 = 3⅓, not a whole number).

Q8. No. 135° × 2 = 270°, which leaves a 90° gap; 135° × 3 = 405°, which is already more than 360°, so the octagons would overlap. Regular octagons do not tessellate by themselves.

Q9. Not true for regular polygons. The interior angle must divide 360°, and at least three must meet at a point. Only equilateral triangles, squares and regular hexagons work. Accept a clear counter-example, such as the regular pentagon (a 36° gap) or the regular octagon.

Q10. A translation is a slide in a straight direction, with no turning. A rotation is a turn about a fixed point. Accept any clear contrast. (A reflection is a flip over a mirror line — some students may mention that too.)

Q11. A reflection (in a vertical mirror line).

Q12. The piece cut from one side is attached, by a translation, to the opposite side. When the finished tile is slid by one side-length, the bump fits exactly into the hole. So the tiles meet with no gaps and no overlaps. Accept any clear version of this idea.

Q13. Yes. You have only moved a piece of the square from one side to the opposite side. Nothing was added or thrown away, so the area is unchanged.

Extension

E1. 90° + 135° + 135° = **360°**. So a square and two regular octagons can meet at a point (the 4.8.8 tessellation).

E2. Many answers are possible. Two reliable ones: a triangle and two regular dodecagons, 60° + 150° + 150° = 360° (written 3.12.12); or triangle, hexagon, triangle, hexagon, 60° + 120° + 60° + 120° = 360° (written 3.6.3.6). Another is 3.4.6.4: 60° + 90° + 120° + 90° = 360°. Accept any correct combination of regular-polygon interior angles that totals 360°, with at least two different polygons.

E3. Checking $n = 3$ to 8 is enough at Foundation: 60°, 90°, 108°, 120°, then 128.6° (heptagon) and 135°. Only 60, 90 and 120 divide 360 with a whole number of polygons of at least 3. For a fuller reason: the number of polygons around a point is 360° / interior angle = $2n / (n − 2)$, and this must be a whole number $k \geq 3$. That forces $n − 2$ to be a positive factor of 4, so $n − 2 = 1$, 2 or 4, hence $n = 3$, 4 or 6. Praise a clear table or a clear factor argument.

The diagrams on this worksheet are original. They do not reproduce any artwork by M.C. Escher.